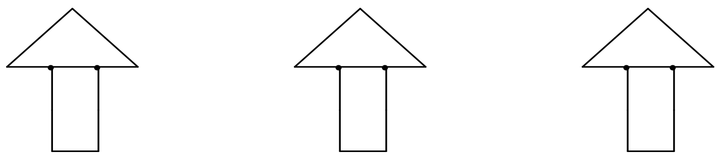
Lecture Note

Advance Graph Theory

Planar and Dual graph

In this section we discuss two important concepts in graph theory – planarity and duality. First we consider planar graphs and derived some properties of these graphs then the dual graphs and related properties. Duality has been of considerable interest to electrical network theories. This interest is due to the fact that the voltages and currents in an electrical network are dual variables.

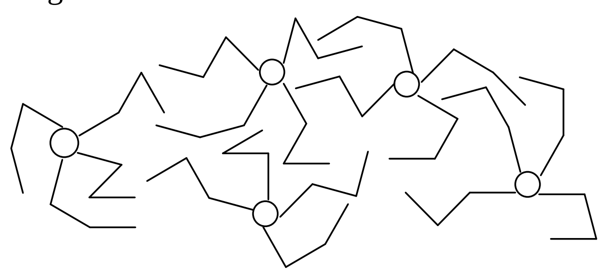
Let us consider a problem with three dogs and three houses.



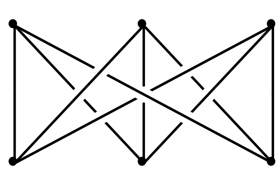
Dog Dog Dog

Can you find a path from each dog to each house such that no two paths intersect?

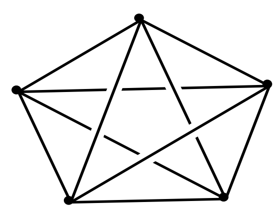
A quadapus is a little-know animal similar to an octopus, but with four arms. Here are five quadapi resting on the sea floor.



Can each quadapus simultaneously shake with others in such a way that no arms cross? Informally, a planner graph is a graph that can be drawn in the plane so that no edges cross. Thus above two examples are asking whether two graphs below are planar, i.e. whether they can be redrawn so that no edges cross.



$K_{3,3}$



K_5

In each case the answer is no. Each drawing would be possible if any one edge is removed.
Now we study to drawing the graphs without crossing on surfaces and deal with some important properties of such graphs.

Embedding: A graph is said to be embeddable in a surface S where it can be drawn on S so that no two edges meet in a point other than their end vertices. A planar graph is one, which can be embedded in the plane. A graph that cannot be drawn on a plane without a crossover between its edges is called non-planar.

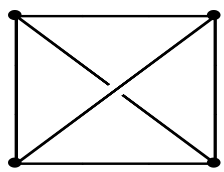


Fig.-1

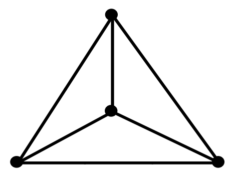


Fig.-2

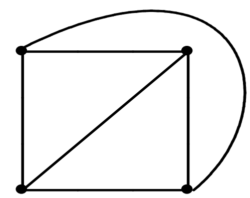


Fig.-3

Here, Fig.-1is not planar but Fig.-2 and Fig.-3are planer graphs.

Theorem: A connected planar graph with n vertices and e edges has e – n +2 regions.

Proof: Here it is sufficient to prove the theorem for a simple graph, because adding a self-loop or a parallel edge simple adds one region to the graph and simultaneously increases the value of e by one. We also remove the edges that do not form the boundaries of any region, because addition of any such edge increase by one and increases n by one, keeping the quantity e – n fixed.
A polygonal net can represent any planer graph, because for simple graphs the plane representation are made by the straight lines. Let the polygonal net representing the given graph consist of r region or faces and let K_p be the p-sided regions.
Since each edge is on the boundary of exactly two regions, we have

$$3.K_3 + 4.K_4 + 5.K_5 + \ldots\ldots + K_s = 2e$$

and $$K_3 + K_4 + K_5 + \ldots\ldots + K_s = r$$

where K_s is the number of polygons with maximum edges.
Now sum of all angles subtended at each vertex in the polygonal net is $2n\pi$.
We know that sum of all interiors and exterior angles of a s-sided region are respectively $\pi(s-2)$ and $\pi(s+2)$. Therefore
$\pi(3 - 2)K_3 + \pi(4 - 2)K_4 + \ldots. + \pi(s - 1)K_s + 4\pi = 2n\pi$ [since grand sum is the sum of interior angles of r – 1 regions and exterior angles of unbounded region].
$\Rightarrow \pi(2e - 2r) + 4\pi = 2n\pi$
$\Rightarrow e - r + 2 = n$
$\Rightarrow r = e - n + 2.$

Corollary 1: If G is a plane (n, e) graph with k components and r regions then n – e + r = k + 1.

Proof: Let $G_1, G_2, \ldots, G_k$ be the components of G each of which is a planar (n_I, e_I) graph. Let r_I denote the number of regions of G_I (i = 1, 2, .. k). Then $r_I = e_I - n_I + 2$ for each i = 1, 2, ….., k. Summing up, we get

$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} (e_i - n_i + 2)$$

$\Rightarrow$ r + (k – 1) = e - n +2k [since unbounded region is counted k – 1 times in the sum]

$\Rightarrow$ n – e + r = k + 1.

Corollary 2: If G be a (n, e) planar graph and every region is an p-cycle, then $e = \dfrac{p(n-2)}{p-2}$, p > 2.

Proof: Let r be the number of regions of G. Since every region is bounded by p edges (including the unbounded region) and each edge lies on the boundary of exactly two regions we have

$$2e = \sum_{i=1}^{r} \text{the number of edges in the boundary of ith region} = rp$$

$$\Rightarrow \frac{2e}{p} = r = e - n + 2$$

$\Rightarrow 2e = p(e - n + 2)$

$\Rightarrow (p - 2)e = p(n - 2)$

$$\Rightarrow e = \frac{p(n-2)}{p-2}.$$

Corollary3: If G be a simple planar (n, e) with regions then $e \geq \frac{3}{2}r$ and $e \leq 3n - 6$, where $n \geq 3$.

Proof: Suppose G is a tree. Then r = 1 and e = n – 1 where n ≥ 3. Therefore

$e \geq \frac{3}{2}r$. Using this inequality in Euler's formula we get $e \leq 3n - 6$.

If G is not a tree then G contains cycles. Hence every edge is on the boundary of at most two regions. Therefore,

$$2e \geq \sum_{i=1}^{r} \text{the number of edges in the boundary of ith region}$$

$\Rightarrow 2e \geq 3r$ [since each region is bounded by at least three edges]

$$\Rightarrow e \geq \frac{3}{2}r.$$

Substituting for r in Euler's formula we get $e \geq \frac{3}{2}(e - n + 2) \Rightarrow e \leq 3n - 6$

Corollary 4: Kuratowski's two graphs K_5 and $K_{3,3}$ are non-planar.

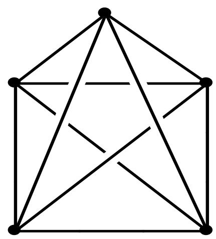


Fig. – 4

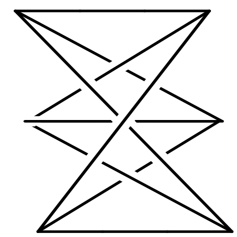


Fig. – 5.

Proof: K_5 is the complete graph of five vertices. Here, $n = 5$ and $e = \frac{n(n-1)}{2} = 10$.

Therefore, $3n - 6 = 15 - 6 = 9 < e(= 10)$.

Thus the graph violates the second inequality of Corollary-3 and hence it is non-planar.

Similarly for $K_{3,3}$ $n = 6$ and $e = 3.3 = 9$

Therefore, $3n - 6 = 16 - 6 = 12 > e\ (= 9)$.

Thus $K_{3,3}$ satisfies the above inequalities, but the graph is non- planar.

To prove the non-planarity of Kuratowski's second graph, we use the additional condition that no region in $K_{3,3}$ can be bounded with fewer than four edges. Hence, if this graph is planar then $2e \geq 4r$ and substituting for r from Euler's formula

$$e \geq 4(e - n + 2)$$

$\Rightarrow 2.9 \geq 4(9 - 6 + 2)$

$\Rightarrow 18 \geq 20$, which is a contradiction.

Duality:

In this section we characterized the class of graphs which have duals. First we prove that every planner graph has a dual. The proof is based on a procedure for constructing a dual of a given planer graph.

Critical Planar Graph:

A graph G is critical planar if G is non-planer but any subgraphs obtain by removing a vertex is planer. The complete graphs K_5 and $K_{3\times3}$ are critical planer.

Geometric Dual:

Given a planar graph G, its geometric dual G* is constructed as follows: Place a vertex in each region of G including the unbounded region. For each edge e of G, draw an edge e* joining the vertices representing the face or faces for which e is a boundary such that e* crosses only the edge e. The new graph G* is called the geometric dual of G. G* is always a plane connected graph.

In figure 1, the graph G with continuous lines, has the graph G* with doted line as its 0dual.
The geometric dual dependents on the particular embedding of the graph in the plane. For example, the graph G is isomorphic to G_1. Whereas G_1* is not isomorphic to G* because the degree of v in G_1* is 6 while there is no vertex of degree 6 in G*.

The following observations can be made about the relationship between a planar graph and its dual:

1. If G is connected then G** is isomorphic to G.
2. A pendent edge of G yields a self-loop in G* and conversely.
3. Edges that are in series in G produce parallel edges in G* and conversely.
4. If n, e, f, r and μ denote the numbers of vertices, edges, regions, rank and nullity of a connected planer graph G, and if n*, e*, f*, r*, and μ* are the corresponding numbers in dual graph G*, then

   $n^* = n, e^* = e, f^* = f.$

Using above relation $r^* = n^* - k^* = f - 1 = e - n + 2 - 1 = e - n + 1 = e - n + k = \mu$ [since G has only one component]

   Similarly $\mu^* = r$.

5. The degree $d_G(f)$ of a face in a planer graph is the number of edges in its boundary, pendant edges being counted twice. In figure-2 $d_G(f_1) = 6$, $d_G(f_2) = 3$.

In general we have $d_{G^*}(f^*) = d_G(f)$.

Theorem: Every planar graph has a dual.

The question arises here that whether a nonplanar graph has a dual. The answer is ‘no’ and it is based on the following two lemmas.

Lemma-1: $K_{3,3}$ has no dual.

Proof: It is observe that

(i) $K_{3,3}$ has no cutsets of two edges.
(ii) $K_{3,3}$ has circuits of length four or six only.
(iii) $K_{3,3}$ has nine edges.

Suppose $K_{3,3}$ has a dual graph G. Then the following observations would be made:

(i) G has no circuits of two edges; i.e. G has no parallel edges.
(ii) G has no cutest with less than four edges. Thus every vertex in G is of degree at least four.
(iii) G has nine edges.

The first two of the above implies that G has at least five vertices each of having degree at least four. Thus G has at least $\frac{1}{2} \times 4 \times 5 = 10$ edges. This contradicts (iii). Hence $K_{3,3}$ has no dual.

Lemma-2: K_5 has no dual.

Proof: It is observe that

(i) K_5 has no circuits of length one or two.

(ii) K_5 has cutsets with four or six edges only.

(iv) K_5 has ten edges.

Suppose K_5 has a dual graph G. Then by (ii), G has circuits of lengths four and six only, i.e. all circuits of G are of even length. So G is bipartite. Since a bipartite graph with six or fewer vertices cannot have more than nine edges, it is necessary that G has seven vertices. But by (i) the degree of every vertex of G is at least three. Hence G must have $\frac{1}{2} \times 7 \times 3 > 10$ edges. Hence K_5 has no dual.

Theorem: A graph has a dual if and only if it is planar.

Proof: The sufficient part of this theorem is the previous theorem. Here we prove the necessary part by showing that a nonplanar graph G has no dual. By Kuratowski's theorem, G has a subgraph H which is homomorphic to $K_{3,3}$ or K_5. If G has dual, then H also has a dual. But then by above theorem $K_{3,3}$ or K_5 should have a dual. This contradict the fact that neither of these graphs have a dual. Hence G has no dual.

Theorem: If G is a planar graph, then $\sum_{f \in F(G)} d(f) = 2e$ where F(G) is the set of all faces of G.

Proof: $\sum_{f \in F(G)} d(f) = \sum_{f^* \in V(G^*)} d(f^*) = 2e(G^*) = 2e.$

Exercise:

1. Explain planarity and dual of a graph.
2. A connected planner graph has nine vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4, and 5. How many edges are there? How many regions are there?
3. Show that adding or deleting loops, parallel edges do not affect the planarity of a graph.
4. Show that any graph with four or fewer vertices is planer.
5. Show that any graph having five or fewer vertices and a vertex of degree 2 is planer.
6. What is the maximum number of edges possible in a planer graph with eight vertices? Ans(18)
7. Show that a planar graph G is self-dual if it is isomorphic to its dual.
8. State and prove Euler's formula for planer graph.
9. Prove that every simple planar graph has at least one vertex of degree less that equal to 5. If such a graph has 11 or less vertices then prove that one vertex has degree less than or equal to 4.
10. Show that complete bipartite graphs K_5 and $K_{3\times3}$ are non-planar.
11. Show that K_5 - e is planar for any edge e of K_5.
12. Show that $K_{3\times3}$ - e is planar for any edge e of $K_{3\times3}$.
13. Show that the complete tripartite graph $K_{1,2,3}$ is non-planer.

14. Show that the wheel graph W_n on n vertices is isomorphic to its dual.
15. Show that a planar connected graph with more than 10 vertices has a complement that is non-planer.
16. Show that a complete graph of 4 vertices is self-dual.
17. Show that if a simple graph G has 11 or more vertices, then either G or its complement $\overline{G}$ is not planer.
18. Show that a graph G is self dual if $|E|$ = 2n – 2, where n is the number of vertices in G.

Graph Coloring

The intuitive idea behind the colouring of a graph is simply paint its vertices such that no two adjacent vertices have same color. What is the minimum number of colors necessary? This constitutes a coloring problem.

After the vertices have been painted, they can be categorized into the groups or its; for example, one set having all the blue colors, another of red and so forth. This is a partitioning problem. The colouring and partitioning can either be carried out on vertices or on edges of the graph and for planer graphs; these can be performed in the regions. Colouring and petitioning have several important applications such as; coding theory, partitioning of logic in digital computer, scheduling and assignments etc.

Definition 1. A colouring of a simple graph is the assignment of a colour to each vertex of the graph so that no two adjacent vertices are assigned the same colour.

A graph can be coloured by assigning a different colour to each of its vertices. However, for most graphs, a colouring can be found that uses fewer colours than the number of vertices in the graph. A colour with colours such that no two adjacent vertices have the same colour is called properly coloured graph.

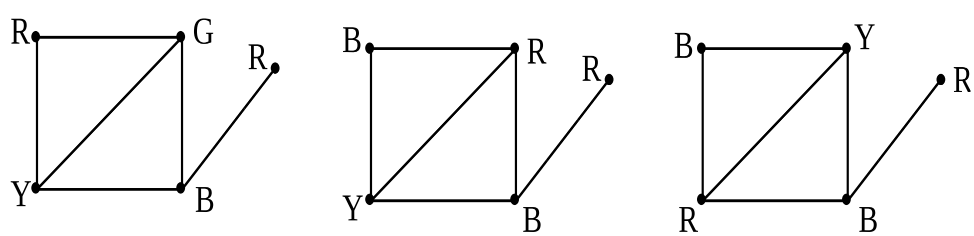


Fig.-1

Definition 2: A graph G is said to be k-colourable if each vertex can be assigned one of k colours so that adjacent vertices get different colours.

Here, we are interested with the proper colouring that requires the minimum number of colours. The minimum colouring of a graph is generally difficult to compute, but the following theorem provides an upper bound.

Theorem 1: A graph G with maximum degree at most k is (k+1)-colourable.

Proof: We use the induction principle on the number of vertex n of G. Let P(n) be the proposition that an n-vertex graph with maximum degree at most k is (k+1)-colorable. A 1-vertex graph G has maximum degree 0 and is 1-colorable, so P(1) is true.
Let us assume that P(n) is true and G be an (n+1) vertex graph with maximum degree at most k. Remove a vertex v from G. The maximum degree of G-v is at most k and G-v is (k+1)-colorable by our assumption P(n). Now add back the vertex v to G. We can assign v a different color from all adjacent vertices, since v has degree at most k and k+1 color are available. Therefore, G is (k+1)-colorable. The theorem follows by induction.

Definition 3: The chromatic number of a graph G denoted by $\chi(G)$ is the least number of colors needed for its proper coloring. Such a graph is called χ-chromatic.

It is easily verified that the graph in Fig.(i),(ii) and (iii) are all 3-chromatic.

In coloring graphs we are concentrates with colouring of connected graphs only (as in disconnected graphs, colouring of the vertices in one component does not effect the other). Self-loops are also not considered for the colouring. Parallel edges between two vertices can be replaced by a simple edge without effecting adjacency of vertices. Thus, for colouring problems we need to consider only simple connected graphs.

Some important observations:

(i) A graph consisting of only isolated vertices is 1-chromatic.

(ii) A graph with one or more edges (without a self-loop) is at least 2-chromatic.

(iii) $\chi(K_n) = n$ where K_n is the complete graph with n vertices.

(iv) $\chi(K_{m,n}) = 2$ where $K_{m,n}$ is the complete bipartite graph with m, n vertices.

(v) $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$ where C_n is the cycle graph with n vertices.

(vi) $\chi(W_n) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$ where W_n is the wheel graph with n vertices.

In general, it is extremely difficult to determine the chromatic number of a graph. For graphs with a small number of vertices it is not too difficult to guess the chromatic number.
Here, we give an algorithm by Welch and Powell for coloring of a graph G. We emphasize that this algorithm does not always yield a minimum coloring of G.

Algorithm: (Welch and Powell)

Let G is a simple connected graph.

Step-1. Order the vertices of G according to decreasing degrees.

Step-2. Assign the first color c_1 to the first vertex and then, in sequential order, assign c_1 to each vertex, which is not adjacent to a previous vertex.

Step-3. Repeat step-2 with a second color c_2 and the subsequence of non-color vertices.

Step-4. Repeat step-3 with a third color c_3, then a forth color c_4 and so on until all vertices are colored.

Step-5. Stop.

Example: Consider the graph G in the following figure. Ordering the vertices according to decreasing degree yields the following sequence:

v_1 , v_4 , v_7 , v_2 , v_3 , v_5 , v_6 , v_8.

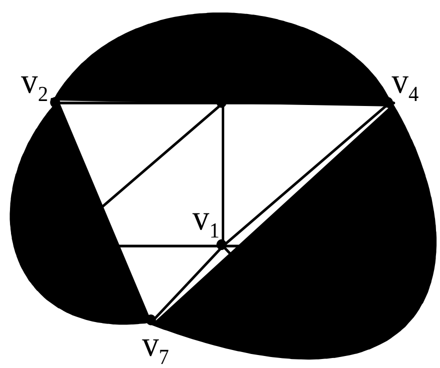


Fig.-2

The first color is assigned to vertex v_1 and v_2.
The second color is assigned to vertex v_4, v_5 and v_8.
The third color is assigned to vertex v_7, v_3 and v_6.
All the vertices have been assigned a color and so G is 3-colorable. Observe that G is not 2-colorable since vertices v_2, v_3, v_4 that are connected to each other, must be assigned different colors. Accordingly $\chi(G) = 3$.

Theorem 2: Every tree with two or more vertices is 2-chromatic.
Proof: Chose any vertices v in the given tree T. Let T be a rooted tree at vertex v. suppose the first color is assigned to the root v. Paint all the vertices adjacent to v with second color. Next paint the vertices adjacent to this using first color. Continue this process till every vertex in T has been painted. Hence all the vertices at odd distance from v have second color. While v and vertices at even distances from v have first color.

Theorem 3: A graph is 2 chromatic if and only if it is bipartite.

Proof : Let G be a 2 chromatic graph, i.e $\chi(G) = 2$.
Let V_1 denote the set of all vertices for which first color is assigned and V_2 be the set of all vertices for which second color is assigned. Then $V = V_1 \cup V_2$ is a partition of V in G. Otherwise at least two vertices in V_1 or V_2 have the same color. Therefore, G is bipartite.
Conversely, let us assume that G is bipartite. Let (V_1, V_2) be the partition of V in G. Then a 2 coloring for G can be given by coloring the vertices in V_1 by one color and the reaming vertices in V_2 by another color. Hence G is 2 chromatic.

Theorem 4: For any graph G, $\chi(G) \leq 1 + \Delta(G)$ where $\Delta(G)$ is the maximum degree of a vertex in G.

Proof: Let H be the smallest induced subgraph of G such that $\chi(H) = n$. Then $\chi(H - v) = n - 1$ for any vertex $v \in H$. It follows that $d(v) \geq n - 1$ so that $n - 1 \leq d_{max}(H) \leq d_{max}(G) = \Delta(G)$.

Five color theorem of a planar graph:

Theorem (Heawood) 5: Every planar graph is 5-colorable.

Proof: Let G be a simple connected planer graph with n-vertices. If $n \leq 5$, the result is obvious. Let $n > 5$. We complete the proof by induction hypothesis. Since G is a connected planer graph. It must have a vertex of degree less than or equal to 5. Let v be a vertex of G such that $d(v) \leq 5$. Let $G' = G - v$. Then by induction hypothesis G' is 5-colorable. When v has degrees 1, 2, 3 and 4 there is no difficulty, since we can give to v one more color.
We are left with the case when $d(v) = 5$ and all the five colors are used in coloring vertices v_1, v_2, v_3, v_4, v_5 which are adjacent to v in G. Let v_i be colored by c_i (say) for i =1, 2, 3, 4 and 5.

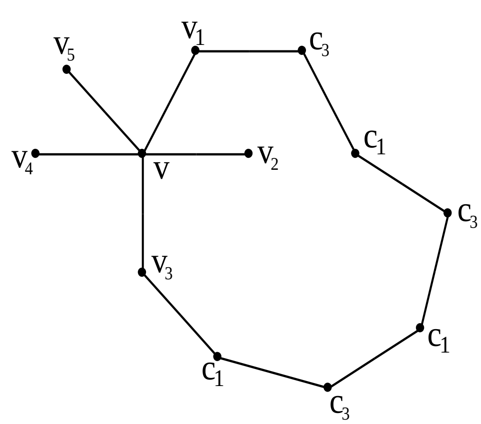


Fig.-3

Suppose there is a path between the vertices v_1 and v_3 colored alternately with color c_1 and c_3 as shown in fig. 1. Then, a similar path between v_2 and v_4 cannot exist, since this path, if it exist will intersect the path between v_1 and v_3, contradicting that G is planer. We can interchange the colors of all vertices connected to v_2, giving color c_4 to v_2, while v_2 has still color c_4. Thus we can point v with color c_2 and hence the graph $G= G^/ + v$ is 5-colorable. Similarly, if we have assumed that no path exists between v_1 and v_3, we could paint v_3 with color c_1 and v with color c_3 and we set a 5-coloring of G.

Theorem (Four-color theorem) 6: The chromatic number of a planer graph is no greater than four. Coloring the vertices of the geometric dual of a planar graph is equivalent to the coloring of the regions of the map, each vertex inheriting the color of the region, to which it belongs and vice versa. in this set-up , the four color conjecture states that every planar graph is four colorable. The conjecture appeared in an article by Alfred Kempe (1879). The problem was originally posed by Guthrie in 1852. it was finally proved by the American mathematician Kenneth Apple and Wolfgang Haken in 1976. The proof depends on a substantial amount of computer calculations. A computer free proof of the theorem is unknown. Apple and Haken showed that if the four color theorem was false, then there must be a counter example among one of approximately 2000 different types of planer graph and they then showed that none of these types could lead a counter example. They used over 1000 hours of computer time in their proof. This proof generated a large amount of controversy, since computer played such an important role in it. For example, could there be an error in a computer program that led to in correct results. Was their argument really a proof if it dependent on what could be unreliable computer output?

Note: Every graph with e edges satisfies $\chi(G) \leq \frac{1}{2}+\sqrt{2e+\frac{1}{4}}$.

Example : Find the chromatic number of the following graph

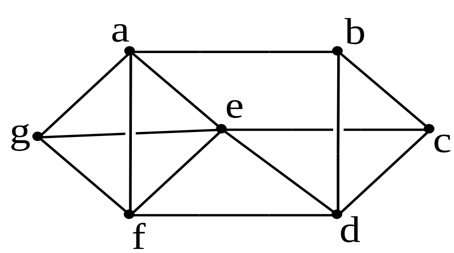


Fig.-4

Solution: $d(b) = d(g) = d(c) = 3$ and
$d(a) = d(d) = d(f) = 4, d(e)=5$.
Using the inequality $\chi(G) \leq 1+ \Delta(G)$
$\chi(G) \leq 5$.

Now since G has a triangle sub graph $3 \leq \chi(G) \leq 5$.
Suppose $\chi(G) = 5$ then G should have 5 vertices with degree at least 4, but there are only 3 vertices in G . $\chi(G) \neq 5$.
Hence $3 \leq \chi(G) \leq 4$.
Now G is not 3 colourable since a, e, g, f which are connected each other, must be assigned different colours.
Therefore $\chi(G) = 4$.

Definition: The edge chromatic number of a graph G is maximum number of colors to color all the edges of G, so that edges with common end points are colored differ colors.
Note that if $\Delta(G)$ is less than or equal to the edge chromatic number of G.

Example:

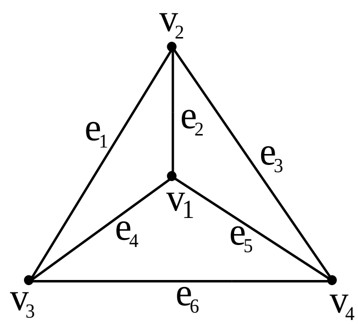


Fig.- 5

Here, edge chromatic number is 3 though $\chi(G) = 4$.

Chromatic polynomial: chromatic polynomials are concerned with description of the property of the proper coloring of a graph. The evaluation or computation of the total number of ways of proper coloring of a graph G of n vertices of using λ or fewer colors is expressed by means of polynomial.

Example: let us consider the following linear graph.

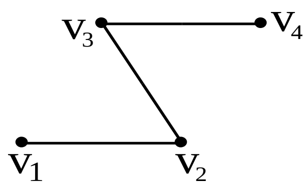


Fig.- 6: Linear Graph with 4 vertices (L_4)

We may color the vertex v_1 with any one of λ colors the second vertex v_2 can be color properly in λ-1 ways. Father we may color the third vertex v_3 using any of λ - 1 color which has not been used in v_2 similarly in v_4.

Hence the fundamental principle of counting the total number of different coloring from λ colors of the graph L_4 is $P_{G_1}(\lambda) = \lambda(\lambda - 1)^3$.

Theorem 1: A n vertex graph is a tree if and only if it's chromatic polynomial

$P_G(\lambda) = \lambda(\lambda - 1)^{n-1}$.[to proved the theorem by indication (example - 1)]

Theorem 2: A graph G of n vertices is complete graph if its chromatic polynomial is

$$P_G(\lambda) = \lambda(\lambda - 1)(\lambda - 2)\ldots\ldots\ldots\ldots\ldots..(\lambda - n + 1).$$

Proof: let V (G) = $\{v_1, v_2, \ldots, v_n\}$. v_1 can be color in λ different ways. Now for each coloring of v_1 there remain λ - 1colors v_2 (adjacent to v_1 because G is complete). Therefore v_2 can be colored in λ - 1 ways. For each coloring of v_1 and v_2, v_3 can be colored in (λ - 2) ways and so on.

Hence $P_G(\lambda) = \lambda(\lambda -1)(\lambda -2)\ldots\ldots\ldots\ldots\ldots..(\lambda - n + 1)$.

Theorem 3: If G is a disconnected graph with components $G_1, G_2, \ldots\ldots\ldots.., G_i$ then the product $P_{G_1}(\lambda), P_{G_2}(\lambda), \ldots\ldots P_{G_r}(\lambda)$, of there respect polynomials equal to $P_G(\lambda)$.

Proof: Left as an exercise.

Theorem 4: If v_1 and v_2 be the two nonadjacent vertices of a graph G then its polynomial $P_G(\lambda) = P_{G_1}(\lambda) + P_{G_2}(\lambda)$. Where G_1 and G_2 are obtain from G by adding an edge between v_1, v_2 and by fusion of v_1, v_2 respectively.

Proof: $P_G(\lambda)$ = the number of different colorings from λ colors = (the number of λ colorings of G in which u and v receive different colors) + (the number of λ coloring in which u and v receive the same color)

= (number of λ coloring of G_1) + (number of λ coloring of G_2)

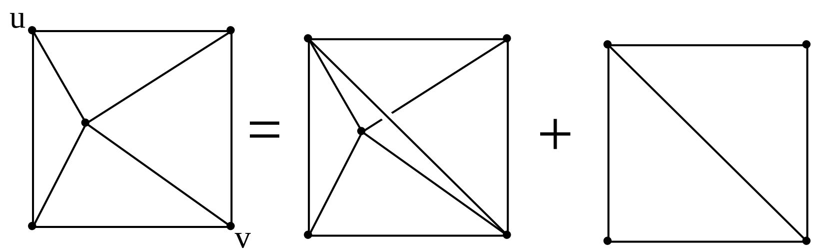
$= P_{G_1}(\lambda) + P_{G_2}(\lambda)$.

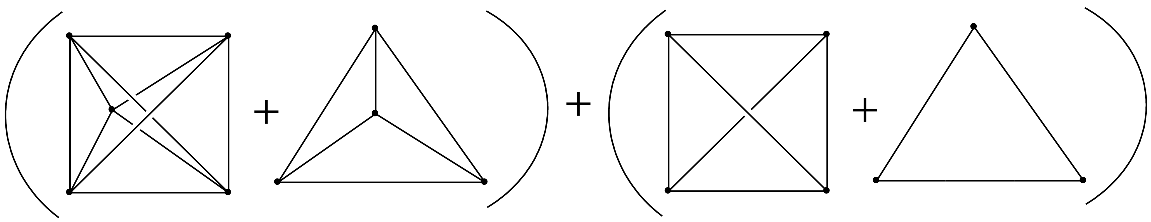
Theorem 5: For any graph G, $P_G(\lambda)$ is a polynomial of degree $|V(G)|$.

Proof: By above theorem $P_G(\lambda) = P_{G_1}(\lambda) + P_{G_2}(\lambda)$ where G_1 has the same number of vertices as G but has one more edge and G_2 has one vertex less than G. containing this process from G_1 and G_2 we will arrive at a finite number of graphs say $H_1, H_2, \ldots\ldots..$, H_m, where each H_i (i = 1,2,……….m) will be complete and $\max|V(H_i)| = |V(G)|$. Since

$P_G(\lambda) = \sum_{i=1}^{m} P_{H_i}(\lambda)$, $P_G(\lambda)$ is a polynomial in λ of degree $|V(G)|$.

Example: the following figure illustrates how the chromatic polynomial of a graph G is as a sum of chromatic polynomial of complete graphs.



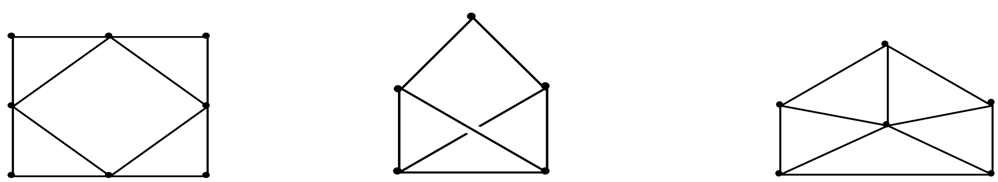


$= (K_5+K_4) + (K_4 + K_3)$
$= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4) + 2\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + \lambda(\lambda - 1)(\lambda - 2)$
$= \lambda(\lambda - 1)(\lambda - 2)(\lambda - 5)(\lambda + 7).$

Theorem 6: For each graph G, the constant term in $P_G(\lambda)$ is zero.
Proof: For each graph G, λ (G) > 0 because the vertex set V(G) of G is not a null set. If $P_G(\lambda)$ has a constant term r, then $P_G(0) = a \neq 0$. This implies that G is properly colorable by 0 colors, which is G contradiction.

Exercise:

1. Prove that chromatic polynomial of any tree with n vertices is $k(k-1)^{n-1}$.
2. Find the chromatic polynomials of the following graphs and hence determine their chromatic numbers.



Graph matching and covering

Within a class of students (containing some girls and some boys), each girl likes some boys and does not like others. Under what conditions can each girl are paired up with a boy that she likes? Many applications of graphs involve such pairings. We can model the situation with a bipartite graph, since it has a vertex partition into two sets. If a girl likes a boy, put an edge between them. Here our goal is to find a matching (or assignment) for the girls. That is, a subset of edges such that exactly one edge is incident to each girl and at most one edge is incident to each boy.

Definition 1: A subset M of the edge set E of a graph G(V, E) is said to be a matching if no two edges in M are incident on the same vertex.

The size of a matching M is the no of edges in M. A matching M is said to be a maximal if there is no matching $M^{/}$ strictly containing M. for example, in a triangle any single edge is a maximal matching. But the graph in fig have two maximal matching (the matching edges are shown in heavy lines) clearly, a graph may have different maximal matching with different sizes. But we are interested in the maximal matching with largest size.

Definition 2: A matching M is said to be a maximum matching of G(V,E) if there is no matching $M^{/}$ such that

$$|M'| > |M| \quad (|M| \text{ is the size of M}).$$

The cardinality (size) of a maximum matching M is denoted by V(G) is called matching number (edge independence number) of G.

Definition 3: if M is a matching in a graph G(V,E) a vertex $v \in V$ is said to be M saturated if there is an edge in M incident on v. Vertex v is said to be M unsaturated if there is no edge in incident on v.

If M saturates all the vertices of G then M is said to be a perfect matching.

Definition 4: If G(V,E) is a bipartite graph with $V=V_1 \cup V_2$ then a matching M of G saturating all vertices in V_1 is called a complete matching.

A complete matching (if it exists) is the largest matching. Whereas the converse is not necessary true.

Definition 5: Given a matching $M \subseteq E$ in a graph G(V, E), an M-alternating path P is a path in G whose edges are alternating in M and out side of M (i.e. if an edge in P is in M then the next edge of P is outside M and vice-versa).

An alternating path P, with respect to M is termed as M-augmenting if its end vertices are not incident with any edge of M (that is end vertices are M-unsaturated).

Proposition 1: replacing $M \cap E(P)$ by $E(P) - M$ produces a new matching $M^{/}$ with one more edge than M.

Theorem 1: A matching $M \subseteq E$ in a graph G(V,E) is a maximum matching if and only if G has on M-augmenting path.

Proof: Assume, for contradiction, that there exists an M-augmenting path P. Then by deleting the edges of $E(P) \cap M$ and adding the edges of $E(P) \cap \overline{M}$ we get a new matching

$M' = (M - E(P)) \cup (E(P) - M)$. It is easy to verify that M' is indeed a valid matching in G. Further $|M'| = |M| + 1$. This contradicts the fact that M is a maximum matching.

Conversely, suppose M is a matching without any M-augmenting path in G. let M′ be any maximum matching in G. By the necessary proof, M′ has no M′-augmenting path in G. Hence the component of $M \Delta M'$ $(= M \cup M' - M \cap M')$ can only be single vertex. This implies that $|M'| = |M|$ and hence M is also a maximum matching.

Hall's marriage problem:

Given a finite set of girl each of whom knows several boys, under what condition can we marry of the girls in such a way that each girl marries a boy he knows. The problem was solved by P. Hall in 1935, and knows as Hall's marriage theorem.
Hall's marriage theorem states necessary and sufficient conditions for the existence of a matching in a bipartite graph. Hall's, theorem is a very useful mathematical tool in graph theory.
Here we proof the Hall's theorem using girl marry-boy terminology. For example the relation between the girls and boys as shown below:

girls	boys known to the girl
g_1	$b_2,\ b_4$
g_2	b_1
g_3	$b_2,\ b_3,\ b_4$
g_4	$b_1,\ b_4,\ b_5$

For us to have any chance at all of matching up the girls, the following 'marriage condition' must hold:

'***Every sub set of girls marry at least as large a set of boys***'.

For example, we cannot find a matching if some 4 girls like only 3 boys. Hall's theorem says that this necessary condition is actually sufficient, if the marriage condition holds, then a matching exist.

Hall's theorem: Let $G = (V_1 \cup V_2, E)$ be a bipartite graph such that $|V_1| \le |V_2|$. Ten G has a complete matching saturating every vertex of V_1 if an only if $|S| \subseteq N(S)$ for every set $S \subseteq V_1$, where N(S) is the neighborhood subset of V_2 that are adjacent to some vertex in S, i.e.

$$N(S) = \{v \in V_2 : \text{there exist } u \in S, \{u, v\} \in E\}.$$

Graph Covering:

In graph theoretic concept, it is natural to say that an edge {u, v} of a graph G covers the vertices u and v. Similarly, a vertex of G is said to covers the edges with which it is incident. From this viewpoint, it is natural to ask, what is the minimum number of vertices (edges) needed to cover all the edges (vertices) of G? One can ask a converse question. What is the maximum number of vertices (edges), which are mutually non-adjacent? Such sets of vertices (edges) are termed as independent vertices (edges).

Covering with vertices and edges are closely related with independent set and matching.

Definition 1: A vertex covering of a graph G(V, E) is a set of vertices $Q \subseteq V(G)$ so that every edge of G is adjacent to at least one vertex in Q. We say Q covers the edges of G.

Definition 2: A vertex covering $Q \subseteq V(G)$ of a graph G is said to be minimal covering if no vertex can be removed without destroying its ability to cover G.

A graph has many minimal coverings and they may be different sizes. The number of vertices in a minimal covering of the smallest size is called vertex covering number and denoted by $\omega_V(G)$.

Example 1: In fig-(a) we mark a vertex covering of size 2 and in Fig-(b) show a vertex covering of size 4. Hence $\omega_V(G) = 2$.

Since no vertex can cover two edges of a matching, the size of every vertex covering is at least the size of every matching. Therefore, a matching and a vertex covering of the same size prove that each is optimal. But such proofs exist bipartite graphs only.

Example 2: In the graph of Fig-(a) we mark a vertex covering of size 2 and show a matching of size 2 in heavy lines. The optimal values are same since the graph is bipartite. But in Fig-(b) the optimal values differ by 1.

Theorem 1: If G is a bipartite graph, then the maximum size of a matching in G equal to the minimum size of a vertex covering of G.

Properties:

1. For any graph G the set of its vertices is trivially a vertex covering of G.
2. Every vertex covering contains a minimal covering.
3. For any graph G(V, E), $\omega_V(G)$ + maximum independent set = number of vertices in V.
4. For a complete bipartite graph $G_{m,n}$ has $\omega_V(G_{m,n}) = \min\{m, n\}$.

Independent sets and Covering: Independent number of a graph is the maximum size of an independent set of vertices. No vertex covers two edges of a matching. Similarly, no edge contains two vertices of an independent set. This yields another dual covering problem, called edge covering. For example, a spanning tree or Hamiltonian circuits in a graph are edge coverings.

Definition 3: An edge covering of a graph G(V, E) is a set of edges $L \subseteq E(G)$ so that every vertex of G is adjacent to at least one edge in L. We say L covers the vertices of G.

An edge covering L in G is said to be minimal covering if no edge of L can be removed without destroying its ability to cover the graph.

The minimum edge covering is the smallest cover. The edge covering number $\omega_E(G)$ for the graph G is the size of the minimum edge covering.

Example 3: In fig.- a graph and two of its minimal covering are shown in heavy lines.

Properties:

1. For any graph G the set of its vertices is trivially a vertex covering of G.
2. An edge covering exist if and only if the graph has no isolated vertex.
3. An edge covering of an n-vertex graph will have at least $\lceil n/2 \rceil$ edges. ($\lceil x \rceil$ denotes the smallest integer not less than x).
4. Every pendant edge in a graph is included in every edge covering of the graph.
5. Every edge covering contains a minimal covering.
6. The set of edges E(G) – L(G) is an edge covering if and only if for every vertex v, the degree of vertex in G(V, E-L) ≤ (degree of vertex v in G(V, E)) – 1.
7. A minimal covering of an n-vertex graph can contain no more than n-1 edges.
8. For a complete bipartite graph $G_{m,n}$ has $\omega_E(G_{m,n}) = \max\{m, n\}$.

Theorem 2: An edge covering L of a graph G is minimal if and only if L contains no path of length three or more.

Objective Questions:

1. How many vertices does a regular graph of degree 4 with 10 edges have? Ans 5

2. For which values of n are the following graphs bipartite?
   (i) K_n (ii) C_n (iii) W_n (iv) Q_n
   ans-a) for all n ≥1, b) for all n ≥3, c) for n =3, for all n ≥ 0

3. For which values of n are the following graphs regular?
   (i) K_n (ii) C_n (iii) W_n (iv) Q_n

4. For which values of m and n is $K_{m,n}$ regular?

5. Let G be an arbitrary graph with n vertices and k components. If a vertex is removed from G, the number of components in the resultant graph must necessarily lie between

(i) k and n (ii) k - 1 and k + 1 (iii) k - 1 and n - 1 (iv) k + 1 and n – k

6. Determine which of the following statements are true.
   (i) A subgraph of a complete graph is complete.
   (ii) A subgraph of a bipartite graph is bipartite.
   (iii) Every subgraph of a planar graph is planar.
   (iv) Any complete graph is regular.

7. Determine which of the following statements are true.
   (i) Every Eulerian graph is Hamiltonian.
   (ii) Every Hamiltonian graph is Eulerian.
   (iii) The Peterson graph is not Eulerian.
   (iv) The Peterson graph is not Hamiltonian.

8. Determine which of the following statements are true.
   (i) The Peterson graph is nonplanar.
   (ii) K_5 is nonplanar.
   (iii) $K_{3,3}$ is nonplanar
   (iv) Dual graph of a planar graph is nonplanar.

9. Determine which of the following statements are true.
   (i) The chromatic number of K_n is n.
   (ii) The chromatic number of any cycle is 2.
   (iii) The chromatic number of any nontrivial tree is 2.
   (iv) Every planar graph is 4 colorable.

Permutations

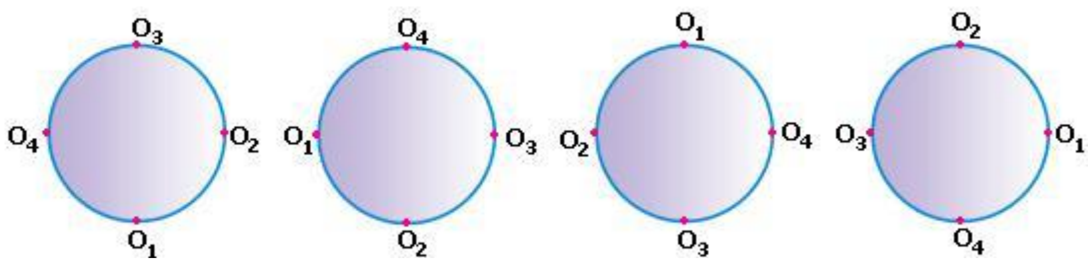
Permutations - The different arrangements that can be made with a given number of things taking some or all of them at a time are called **permutations**. The symbol n P r or P(n,r) is used to denote the number of **permutations** of n things taken r at a time.

permutation is the arrangement of objects with ordering of same and different objects in different ways. Permutation is always greater than combination as in combination we neglect the ordering of the same objects in different ways, so as peter said abve ordering is very important in permutation.

Explain Circular permutation

When things are arranged in places along a line with first and last place, they form a linear permutation. So far we have dealt only with linear permutations. When things are arranged in places along a closed curve or a circle, in which any place may be regarded as the first or last place, they form a circular permutation.
The permutation in a row or along a line has a beginning and an end, but there is nothing like beginning or end or first and last in a circular permutation. In circular permutations, we consider one of the objects as fixed and the remaining objects are arranged as in linear permutation.

Thus, the number of permutations of 4 objects in a row = 4!, where as the number of circular permutations of 4 objects is (4-1)! = 3!.
The following arrangements of 4 objects O1, O2, O3, O4 in a circle will be considered as one or same arrangement.



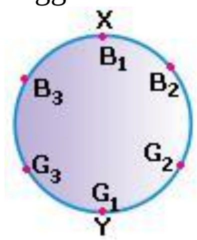
Observe carefully that when arranged in a row, O1 O2 O3 O4, O2O3O4 O1, O3O4O1O2, O4O1O2O3 are different permutations. When arranged in a circle, these 4 permutations are considered as one permutation.

Theorem:
The number of circular permutations of n different objects is (n-1)!.

Proof:
Each circular permutation corresponds to n linear permutations depending on where we start. Since there are exactly n! linear permutations, there are exactly permutations. Hence, the number of circular permutations is the same as   (n-1)!.

Example 1:
1. Three boys and three girls are to be seated around a table in a circle. Among them, the boy X does not want any girl as neighbour and girl Y does not want any boy as neighbour. How many such arrangements are possible?

Suggested answer:



The arrangement is as shown in the figure, the boy X will have B_2, B_3 as neighbours. The girl Y will have G_2, G_3 as neighbours. The two boys B_2, B_3 can be arranged in two ways. The two girls G_2, G_3 can be arranged in two ways.
Hence, the total number of arrangements = 2 x 2 = 4.

Example 2:
In how many different arrangements can 6 gentlemen and 6 ladies sit around a table if

i) there is no restriction and
ii) no two ladies sit side by side?

Suggested answer:
i) Here, the total number = 6 + 6 = 12.

12 persons can be arranged in circular permutation as (12 - 1)! = 11! ways.
ii) When 6 gentlemen are arranged around a table, there are 6 positions, each being between two gentlemen for 6 ladies, when no two ladies sit side by side. Now, the number of ways in which 6 gentlemen can be seated around a table = (6 - 1)! = 5!.

Then, corresponding to each seating arrangement for the gentlemen, the 6 ladies can be seated in 6! ways.
The required number of arrangements = (5!)(6!)

= (120)(720) = 86400

Probability- Introductory idea

1.1 Probability: Definition and example

The word **probability** literally denotes 'chance', and the theory of probability deals with laws governing the chances of occurrence of phenomena which are unpredictable in nature. We shall start with a problem.

1.1.1 Problem:

A spinner has 4 equal sectors colored yellow, blue, green and red. What are the chances of landing on yellow after spinning the spinner? What are the chances of landing on red?
Solution: The chances of landing on yellow are 1 in 4, or one fourth. The chances of landing on red are 1 in 4, or one fourth.

Let's look at some definitions and examples from the problem above.

Definition	Example
The word experiment is used to describe an act which can be repeated under some given conditions.	In the problem above, the experiment is spinning the spinner.
An outcome is the result of a single trial of an experiment.	The possible outcomes are landing on yellow, blue, green or red.
An event is one or more outcomes of an experiment.	One event of this experiment is landing on yellow.
Probability is the measure of how likely an event is.	The probability of landing on red is one fourth.

1.1.2 Random experiment:

Random experiments are those experiments whose results depend on chance such as when a coin is tossed, either Head or Tail appears. But the result of any toss cannot be predicted in advance and is said to 'depend on chance'.

1.1.3 Some Important terminologies:

(1) Mutually Exclusive: Events are said to be 'mutually exclusive', when two or more of them cannot occur simultaneously.
E.g. In tossing a unbiased coin, the elementary events 'Head' and 'Tail' are mutually exclusive because in any toss either Head occurs or Tail occurs; and the occurrence of Head as well as Tail in any toss is impossible.
(2) Exhaustive: Several events are said to form an exhaustive set, if at least one of them must necessarily occur.
Ex : In tossing a unbiased coin , the two events 'Head' and 'Tail' form an exhaustive set; because one of these two must necessarily occur.
(3) Equally Likely: The outcomes of a random experiment are said to be 'equally likely', if after taking into consideration all relevant evidence , none of them can be expected in preference to another.
Ex: If two unbiased coins are thrown, the elementary events HH, HT, TH and TT are equally likely.

In order to measure probabilities, mathematicians have devised the following formula for finding the probability of an event.

$$\text{Probability of an event A} = P(A) = \frac{\text{The number of ways event A can occur}}{\text{The total number of possible outcomes}}$$

1.1.4 Experiment:

A single 6-sided unbiased die is rolled. What is the probability of rolling an even number? Or of rolling an odd number?

Solutions: The possible outcomes of this experiment are 1, 2, 3, 4, 5 and 6.Rolling an even number (2, 4 or 6) is an event, and rolling an odd number (1, 3 or 5) is also an event.

$$P(even) = \frac{\text{ways to roll an even number}}{\text{total number of sides}} = \frac{3}{6} = \frac{1}{2}$$

$$P(odd) = \frac{\text{ways to roll an odd number}}{\text{total number of sides}} = \frac{3}{6} = \frac{1}{2}.$$

1.1.5 Remark:

The probability of an event A, symbolized by P(A), is a number between 0 and 1, inclusive, that measures the likelihood of an event in the following way:

If P(A) > P(B) then event A is more likely to occur than event B.

If P(A) = P(B) then events A and B are equally likely to occur.

1.1.6 Experiment:

A glass jar contains 6 red, 5 green, 8 blue and 3 yellow marbles. If a single marble is chosen at random from the jar, what is the probability of choosing a red marble? A green marble? A blue marble? A yellow marble?

Outcomes: The possible outcomes of this experiment are red, green, blue and yellow. So we can find the probabilities as below

$$P(\text{red}) = \frac{\text{number of ways to choose red}}{\text{total number of marbles}} = \frac{6}{22} = \frac{3}{11}$$

$$P(\text{green}) = \frac{\text{number of ways to choose green}}{\text{total number of marbles}} = \frac{5}{22}$$

$$P(\text{blue}) = \frac{\text{number of ways to choose blue}}{\text{total number of marbles}} = \frac{8}{22} = \frac{4}{11}$$

$$P(\text{yellow}) = \frac{\text{number of ways to choose yellow}}{\text{total number of marbles}} = \frac{3}{22}$$

The outcomes in this experiment are not equally likely to occur. We are more likely to choose a blue marble than any other color. We are least likely to choose a yellow marble.

1.1.7 Experiment:

What is the probability that all three children in a family have different birthdays? (Assume, 1 year = 365 days).

Answer: The 1^{st} child may be born on any of 365 days of the year, the 2^{nd} also on any of the 365 days and similarly the 3^{rd} child. Hence, the total number of possible ways in which the 3 children have birth days $365 \times 365 \times 365$. These cases are mutually exclusive, exhaustive and equally likely.

As regards the number of favorable cases out of these, we note that the 1^{st} child may have any of the 365 days of the year as its birthday. In order that the 2^{nd} child has a birthday different from that of the 1^{st} ,it should have been born on any of the 364 remaining days of the year ;similarly the 3^{rd} should be born on any of the remaining 363 days. So, the number of cases favorable to the event "different birthdays" is $365 \times 364 \times 363$. Hence the required probability is $p = \frac{365 \times 364 \times 363}{365 \times 365 \times 365}$

1.1.8 Conditional probability:

The conditional probability of an event B in relationship to an event A is the probability that event B occurs given that event A has already occurred. The notation for conditional probability is P(B|A), read as the probability of B given A.

1.1.9 Multiplication rule:

When two events, A and B, are dependent, the probability of both occurring is:

$$P(A \cap B) \text{ or } P(\text{ A and B }) = P(A).P(B/A)$$

$$i.e. \quad P(B/A) = \frac{P(A \cap B)}{P(A)}.$$

Now we can use this formula to solve the following problem.

1.10 Problem:

A math teacher gave her class two tests. 25% of the class passed both tests and 42% of the class passed the first test. What percent of those who passed the first test also passed the second test?

Solution: We have to find the probability of passing 2nd test on the basis they passed the first test is $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.25}{0.42} = 0.6$, where A denotes the event qualifying first test, and B denotes the event qualifying 2nd test.

1.1.11 Problem:

Among twenty-five articles, nine are defective, six having only minor defects and three having major defects. Determine the probability that an article selected at random has major defects given that it has defects.

Solutions: Hint. Let MD-major defects, and D-defects

Then we have to find P[MD/D] = $\frac{P(MD \cap D)}{P(D)}$=(3/25)/(9/25)

= 3/9 = 1/3.

1.2 Total probability and example:

1.2.1 Theorem:

(i) If two events A and B are mutually exclusive, then the probability of occurrence of either A or B is given by the sum of their probabilities, i.e.

Probability of (A or B) = Probability of A + Probability of B .

In symbols**,**

$$\mathbf{P(A + B) = P(A) + P(B).}$$

Proof : Let us suppose that a random experiment has n possible outcomes, which are mutually exclusive ,exhaustive and equally likely .If m_1 of these cases are favorable to the event A and m_2 cases are favorable to the event B, then the probability of these events are $P(A)=\frac{m_1}{n}, \quad P(B)=\frac{m_2}{n}$.

Since the events A and B are mutually exclusive (i.e. both of them cannot occur simultaneously) .The number of cases favorable to 'either A or B is therefore m_1+m_2.

Hence $P(A+B)=\frac{m_1+m_2}{n}=\frac{m_1}{n}+\frac{m_2}{n}=P(A)+P(B)$, (Proved)

Theorem of total probability can be extended to any number of mutually exclusive events.

(ii) The probability of the event complementary to A is given by

$$P(\bar{A})=1-P(A).$$

(iii) The probability of occurrence of at least one of the two events A and B (which may not be mutually exclusive), is given by

$$P(A+B)=P(A)+P(B)-P(AB)$$

Proof: Events $A\cap\bar{B}$ and A∩B are mutually exclusive, then A = (A∩ $\bar{B}$)U(A∩B)

=> P(A) = P(A∩ $\bar{B}$) + P(A∩B)

=> P(A∩ $\bar{B}$) = P(A) – P(A∩B)

Similarly, event B is the union of mutually exclusive events A∩B and Ā∩B, and

P(B) = P(A∩B) + P(Ā∩B) => P(Ā∩B) = P(B) – P(A∩B)

Events A∩ $\bar{B}$, A∩B, Ā∩B are mutually exclusive, and their union is the events AUB.

Hence, we have

P(AUB) = P(A∩ $\bar{B}$) + P(A∩B) + P(Ā∩B)
= [P(A) – P(A∩B)] + P(A∩B) + [P(B) – P(A∩B)]
= P(A) + P(B) – P(A∩B).

1.2.2 Experiment:

A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace?

Solution: The sample space S of drawing a card from a well-shuffled pack of playing cards consists of 52 sample points.

If A and B denote the events of drawing a 'spade card' and 'an ace' respectively then A consists 13 sample points and B consists of 4 sample points. So that,

$$P(A)=\frac{13}{52} \text{ and } P(B)=\frac{4}{52}.$$

The compound event A∩B consists of only one sample point, i.e. ace of spade, so that,

$$P(A\cap B)=\frac{1}{52}$$

The probability that the card drawn is either a spade or a ace is given by

$$P(AUB) = P(A) + P(B) – P(A\cap B)$$
$$=\frac{13}{52}+\frac{4}{52}-\frac{1}{52}=\frac{4}{13}$$

1.2.3 Problem:

Try to solve the above problem using the definition of probability.

1.2.4 Problem:

Two unbiased coins are tossed. What is the probability of getting at least one tail?

Answer: Let A denote the event 'getting no tail'. Now, we require to find $P(\bar{A})$.

$$P(A) = \frac{\text{number of favorable cases}}{\text{total number of cases}} = \frac{1}{4}$$

So, $P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{4} = \frac{3}{4}$.

1.3 Model Questions:

- **Objective Questions (True/False Type):**

1. Probability of an event can be more than 1. (T/F)
2. In tossing an unbiased coin, the probability of getting either head or tail is 1. (T/F)

- **Objective Questions (Multiple Choice Type):**

1. A die is rolled. The probability of getting multiple of 2 is

   (a) $\frac{1}{2}$ (b) $\frac{1}{6}$

   (b) $\frac{1}{3}$ (d) none

2. A card is drawn from a well shuffled pack. The probability that it is an ace is

   (a) $\frac{1}{4}$ (b) $\frac{1}{13}$

   (b) $\frac{2}{13}$ (d) none

3. If n dices are rolled then the total number of favorable cases is

   (a) 1/36 (b) 6^n

   © 1/n (d) none

- **Short answer Type Questions:**

1. Two dices are rolled. What is the probability of getting the sum of the two faces is10? (**Hint**. The favorable cases are three, (6, 4); (4, 6); (5, 5) and total no. of cases is 6^2=36.)
2. Two unbiased coins are tossed. What is the probability of getting at least on